HIGHER GALOIS DESCENT FOR CHROMATIC LOCALIZATIONS OF K-THEORY

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1. Intro

The main source for this talk is [BCSY23], and all results cited with no other source specified (e.g. "from Proposition 2.29") are from this paper.

We have two main theorems to prove. The first is that $K_{T(n+1)}$ is p-typically semiadditive. The second is that it preserves certain Galois extensions of $T(n)$-local rings. To prove the first theorem, we will need to use some facts about group algebras and their colimits.

2. Group Algebras

Theorem 2.1 (Corollary 3.6)**.** *We have an equivalence* $\mathrm{Perf}(R[\Omega A]) \simeq \mathrm{Perf}(R)[A]$ *in* $\mathrm{Cat}_{\mathrm{perf}}$*, symmetric-monoidally natural in* $A \in \mathcal{S}_*^{\geq 1}$ *and* $R \in \mathrm{Alg}(\mathrm{Sp})$*.*

To prove this, we'll first prove a fact about colimits. Recall that given a functor $F : \mathcal{C} \to \mathcal{D}$, an object $X \in \mathcal{C}$, and a space A, the universal property of colimits gives us an "assembly map" $F(X)[A] \to F(X[A])$. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are moreover symmetric monoidal, and F is a lax symmetric monoidal functor.

Lemma 2.2. *In this case, the assembly map is canonically lax symmetric monoidally natural in X and A.*

Proof sketch. Let $\widehat{\mathrm{Cat}}_{\mathrm{all}}$ denote the category of cocomplete categories and cocontinuous functors, and let $\mathcal{C}$ be an object of $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{all}})$. There is an obvious symmetric monoidal functor $i : \mathcal{C} \to \mathcal{S} \times \mathcal{C}$ which is the point on the first factor and the identity on the second. Left Kan extension along i is a functor $i_! : \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{S} \times \mathcal{C}, \mathcal{C})$ sending $\mathrm{id}_{\mathcal{C}}$ to the functor $(X, A) \mapsto X[A]$. Because i is symmetric monoidal, this induces an adjunction on lax functor categories, giving $i_!(\mathrm{id}_{\mathcal{C}})$ a lax symmetric monoidal structure; in fact, it is strong, because $\otimes_{\mathcal{C}}$ preserves colimits. This proves our result in the special case $F = \mathrm{id}_{\mathcal{C}}$.

Now, let $F : \mathcal{C} \to \mathcal{D}$ be as given. We can produce a commutative square of lax functor categories, use it to produce a Beck-Chevalley map, and evaluate at $\mathrm{id}_{\mathcal{C}}$ to get the desired assembly map. $\square$

We will also need the following result, which explicitly computes group algebras in Pr^L.

Lemma 2.3. *There is a symmetric monoidally natural equivalence* $\mathrm{LMod}_{\Omega A}(\mathcal{C}) \simeq \mathcal{C}[A]$ *in* Pr^L *for* $A \in \mathcal{S}_*^{\geq 1}$*.*

Date: February 21, 2024.

Proof. These are two symmetric monoidal functors $\mathrm{Pr}^L \times \mathcal{S}_*^{\geq 1} \to \mathrm{Pr}^L$. Products and coproducts coincide for symmetric monoidal categories, so this equivalence can be constructed on each component separately. On the first, they're both the identity functor, so it's enough to prove this for $\mathcal{C} = \mathcal{S}$: $\mathrm{LMod}_{\Omega A}(\mathcal{S}) \simeq \mathcal{S}[A]$. Some abstract nonsense shows that these are both fully faithful when viewed as functors to Pr^L_*, and it is shown in [CSY21b] that they have the same essential image. Since the category of pointed connected spaces has no nontrivial symmetric monoidal auto-equivalences, the functors must be equivalent. □

Corollary 2.4. *We have a symmetric monoidally natural equivalence* $\mathrm{LMod}_{R[\Omega A]}(\mathcal{C}) \simeq \mathrm{LMod}_R(\mathcal{C})[A]$ *in* Pr^L_*. □

Proof of Thm. Take $\mathcal{C} = \mathrm{Sp}$ in the above corollary. We know that $\mathrm{Pr}^L_{\mathrm{st},\omega} \hookrightarrow \mathrm{Pr}^L_{\mathrm{st}}$ preserves colimits, and the categories in question are both in this subcategory. Applying the compact objects functor $(-)^\omega$ gives the desired equivalence. □

3. Semiadditivity of Localized K-theory

We are ready to prove our first main theorem.

Theorem 3.1. *The functor* $K_{T(n+1)} : \mathrm{Cat}_{M_n^f} \to \mathrm{Sp}_{T(n+1)}$ *is p-typically* ∞*-semiadditive; that is, it preserves limits and colimits indexed over* π*-finite p-spaces.*

Proof. We prove this through a series of reductions.

Firstly, because both of these categories are themselves ∞-semiadditive, we only need to prove preservation of colimits.

Secondly, by an earlier result from our study of semiadditivity (Prp 2.29), it is enough to prove this for constant diagrams indexed over Eilenberg-MacLane spaces. This amounts to showing that the assembly map $K_{T(n+1)}(\mathcal{C})[A] \to K_{T(n+1)}(\mathcal{C}[A])$ is an isomorphism for $A = K(G, m)$ where G is a finite p-group. Now, we induct on m.

The case $m = 0$ is easy: this just requires preservation of coproducts, which follows from exactness of $K_{T(n+1)}$. The case $m = 1$ follows from Theorem 4.12 of [CMNN20] with $E = L_n^f\mathbb{S}$. Henceforth, assume $m \geq 2$.

We can reduce to the case $\mathcal{C} \simeq \mathrm{Perf}(R)$, $R \in M_n^f\mathrm{Sp}$: by a corollary of the Schwede-Shipley theorem (Prp 2.9), $\mathcal{C}$ is a filtered colimit of categories of the form $\mathrm{Perf}(R_i)$, where the R_i are endomorphism rings of objects of $\mathcal{C}$ and are thus n-monochromatic. Since $K_{T(n+1)}$ preserves filtered colimits, we may assume $\mathcal{C}$ is of this form.

This is where Corollary 3.6 enters the picture. Since we are assuming $m \geq 2$, both A and ΩA are connected; so by that result, our map is $K_{T(n+1)}(R)[A] \to K_{T(n+1)}(R[\Omega A])$.

Finally, there are two cases to consider. Firstly, suppose $n = 0$. Then R is rational, so $R[\Omega A] \simeq R[*] \simeq R$ because Eilenberg-MacLane spaces of finite groups have no rational cohomology. Also, by a result of [CSY21a], constant A-shaped colimits in $\mathrm{Sp}_{T(1)}$ do not change the object, i.e. $\mathrm{colim}_A X \simeq X$ naturally. So in this case, the assembly map is the identity.

Secondly, we have the $n \geq 1$ case. We use the bar construction to write $A \simeq \mathrm{colim}_{\Delta^{op}} A_k$, where $A_k := (\Omega A)^k$. We have a commutative diagram:

$$\begin{array}{ccc}
\operatorname{colim}_{\Delta^{op}} K_{T(n+1)}(R)[A_k] & \longrightarrow & \operatorname{colim}_{\Delta^{op}} K_{T(n+1)}(R[\Omega A_k]) \\
\Big\downarrow & & \downarrow \\
 & & K_{T(n+1)}(\operatorname{colim}_{\Delta^{op}} R[\Omega A_k]) \\
 & & \downarrow \\
K_{T(n+1)}(R)[\operatorname{colim}_{\Delta^{op}} A_k] & \longrightarrow & K_{T(n+1)}(R[\Omega \operatorname{colim}_{\Delta^{op}} A_k])
\end{array}$$

The top horizontal map is an isomorphism by our inductive hypothesis. The left vertical map is an isomorphism because $K_{T(n+1)}(R)[-]$ preserves colimits. Since $n \geq 1$, a corollary of purity ([LMMT20]) implies $K_{T(n+1)}$ preserves sifted colimits, so the top right vertical map is an isomorphism. Finally, $R[\Omega(-)]$ preserves sifted colimits, so the bottom right vertical map is an isomorphism. It follows from all of this that the lower horizontal map is an isomorphism, which is what we wanted to show. □

There are some cool corollaries of this that I'll present without proof.

Corollary 3.2.

- *This remains true if we replace the domain with* $\mathrm{Cat}_{L_n^f}$.
- *There is a symmetric monoidally natural map* $K_{T(n+1)}(R)[A] \to K_{T(n+1)}(R[\Omega A])$, *which is an equivalence when* A *is a sifted colimit of pointed* π*-finite* p*-spaces.*
- $K_{T(n+1)} : \mathrm{CAlg}(\mathrm{Sp}_{T(n)}) \to \mathrm{CAlg}(\mathrm{Sp}_{T(n+1)})$ *preserves limits indexed by* n*-finite* p*-spaces. In particular,* $K_{T(n+1)}(R^A) \simeq K_{T(n+1)}(R)^A$.
- *For* L_n^f*-local categories with* m*-finite colimits,* $K_{T(n+1)}$ *coincides with* $K^{[m]}_{T(n+1)}$*, the* $T(n+1)$*-localization of the universal* m*-semiadditive version of algebraic* K*-theory.*

4. Higher Galois Descent

Stating our second main theorem will require some definitions.

Definition 4.1. Let $\mathcal{C}$ be symmetric monoidal, and suppose $BG \in \mathcal{S}_*^{\geq 1}$ is weakly $\mathcal{C}$-ambidextrous with diagonal Δ. Write q for the tautological map $BG \to *$ and e for the basepont map $* \to BG$. We say a G-equivariant commutative algebra $R : BG \to \mathrm{CAlg}(\mathcal{C})$ is *Galois* if

i) The map $\mathbb{1} \to q_* R =: R^{hG}$ obtained by transposing the unit is an isomorphism; and

ii) The map $R \otimes R \to \Delta_* R =: R^G$ obtained by transposing multiplication is an isomorphism.

(I will refer to these as the *extension* and *separability* conditions, respectively.)

In the classical case of a finite Galois extension L/K with Galois group G, $\mathcal{C}$ would be Mod_K, R would be L with the Galois action, the extension condition would be the fixed-point isomorphism, and the separability condition would be the isomorphism $L \otimes_K L \cong L^G$.

We can apply the functor $\mathrm{Mod}_{(-)}(\mathcal{C})$ to R to get a G-equivariant $\mathcal{C}$-linear symmetric monoidal category $\mathrm{Mod}_R(\mathcal{C})$, which comes with a symmetric monoidal functor $R \otimes (-) : \mathcal{C} \to \mathrm{Mod}_R(\mathcal{C})^{hG}$. In the classical case, this is an equivalence. As it turns out, it is also an equivalence here provided that R is faithful, meaning $R(*) \otimes M \simeq 0$. I'll skip the proof for time, but suffice to say it involves moving the BG around and studying a mild generalization of the separability isomorphism.

A result from [CMNN20] will prove useful, allowing us to reduce to dualizable objects in proving our second main theorem. Let A be a $T(n)$-local $\mathbb{E}_\infty$-ring, and write $K'(A)$ for the K-theory of the category of dualizable objects in $T(n)$-local A-modules. Because this includes $\mathrm{Perf}(A)$ as a subcategory, we get an induced map $K(A) \to K'(A)$.

Lemma 4.2. *The homotopy fiber of $K(A) \to K'(A)$ is naturally a module over $K(L^f_{n-1}\mathbb{S})$.*

We finally have the language and tools to state and prove our second main theorem.

Theorem 4.3. *Let $n \geq 0$, and let G an n-finite p-group. For every $T(n)$-local G-Galois extension $R \to S$, the induced $T(n+1)$-local G-extension $K_{T(n+1)}(R) \to K_{T(n+1)}S$ is Galois.*

Proof. By Galois descent, we have $\mathrm{Mod}_R(\mathrm{Sp}_{T(n)}) \simeq \mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{hG}$. We can pass to the dualizable objects, so $\mathrm{Mod}_R(\mathrm{Sp}_{T(n)})^{dbl} \simeq \left(\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{hG}\right)^{dbl}$. This operation commutes with taking homotopy fixed points by a result of Lurie (in HA), so $\mathrm{Mod}_R(\mathrm{Sp}_{T(n)})^{dbl} \simeq \left(\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{dbl}\right)^{hG}$. Finally, we have isomorphisms

$$\begin{aligned}
K_{T(n+1)}R &\simeq K_{T(n+1)}(\mathrm{Mod}_R(\mathrm{Sp}_{T(n)})^{dbl}) && \text{by lemma} \\
&\simeq K_{T(n+1)}\left(\left(\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{dbl}\right)^{hG}\right) && \text{by the above} \\
&\simeq K_{T(n+1)}\left(\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{dbl}\right)^{hG} && \text{by our other main theorem} \\
&\simeq K_{T(n+1)}(S)^{hG} && \text{by lemma,}
\end{aligned}$$

which proves the extension condition. A result of [BCSY22] implies that the separability condition is automatic, which concludes the proof. □

References

[BCSY22] Tobias Barthel et al. *The Chromatic Fourier Transform.* Oct. 23, 2022. DOI: 10.48550/arXiv.2210.12822. arXiv: 2210.12822 [math]. preprint.

[BCSY23] Shay Ben-Moshe et al. *Descent and Cyclotomic Redshift for Chromatically Localized Algebraic K-theory.* Sept. 13, 2023. DOI: 10.48550/arXiv.2309.07123. arXiv: 2309.07123 [math]. preprint.

[CMNN20] Dustin Clausen et al. *Descent and Vanishing in Chromatic Algebraic K-Theory via Group Actions.* Version 2. Nov. 8, 2022. DOI: 10.48550/arXiv.2011.08233. arXiv: 2011.08233 [math]. preprint.

[CSY21a] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. *Ambidexterity and Height*. Sept. 25, 2020. DOI: 10.48550/arXiv.2007.13089. arXiv: 2007.13089 [math]. preprint.

[CSY21b] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. *Chromatic Cyclotomic Extensions*. May 24, 2023. DOI: 10.48550/arXiv.2103.02471. arXiv: 2103.02471 [math]. preprint.

[LMMT20] Markus Land et al. "Purity in Chromatically Localized Algebraic K-Theory". In: *Journal of the American Mathematical Society* (Feb. 1, 2024). ISSN: 0894-0347, 1088-6834. DOI: 10.1090/jams/1043. arXiv: 2001.10425 [math].